VIII

FUNCTIONS HAVING ONLY ONE SINGULARITY

40. An important problem in the theory of Taylor's series is that of determining the conditions to be satisfied by the coefficients in order that the function defined by the series (with finite radius of convergence) shall have exactly one singularity in the entire plane. The problem was solved first by Leau.[1] Later Faber[2] established necessary and sufficient conditions. His theorem is the following:

THEOREM 1: *In order that the function*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad 0 < R < \infty,$$

have the point 1 *as its only singularity in the entire plane, it is necessary and sufficient that* $a_n = g(n)$, $n = 1, 2, \cdots$, *where* $g(z)$ *is an integral function such that*

$$|g(z)| < e^{r\epsilon} \textit{ for } r > r_\epsilon, \ r = |z|.$$

Leau gave, as a sufficient condition, $a_n = g(n)$, where

$$|g(z)| < e^{r^{s+\epsilon}}, \text{ for } r > r_\epsilon, \ 0 < s < 1.$$

The latter condition is more restrictive than the former. For we may write

$$e^{r^{s+\epsilon}} < (e^r)^{r^{s+2\epsilon-1}}$$

[1] Journal de Math., 5e série, t. v. (1899), p. 409.
[2] Math. Annalen, t. 57 (1903), p. 369.

and since $s < 1$, we have, for a suitably chosen α,

$$s + 2\epsilon - 1 < -\alpha < 0,$$

if ϵ is sufficiently small. Hence

$$|g(z)| < e^{r r^{-\alpha}} \text{ for } r > r_\epsilon.$$

If we take r'_ϵ greater than each of the numbers r_ϵ, $\left(\frac{1}{\epsilon}\right)^{\frac{1}{\alpha}}$, we have

$$|g(z)| < e^{\epsilon r} \text{ for } r > r'_\epsilon.$$

If, then, the condition of Leau is satisfied, so is that of Faber.

Assume first that it has been shown that there is no singularity except at the point $x = 1$, if the a_n satisfy the conditions of the hypothesis. If, also, $x = 1$ is a regular point, we must have $a_n = g(n) = 0$, $n = 1, 2, \cdots$, since, by Liouville's theorem, $f(x)$ reduces to a constant. Suppose $g(z)$ has a zero of order k at $z = 1$. Then

$$g_1(z) = \frac{g(z)}{(1-z)^k}$$

is also an integral function which satisfies the hypothesis of the theorem, and we have $g_1(n) = 0$, $n = 2, 3, \cdots$. The series $\Sigma g_1(n)x^n$, which reduces to $xg_1(1)$, would be regular at ∞, which is impossible. Hence $x = 1$ is actually a singular point.

We remark that unless g is a polynomial, the point 1 is an essential singularity. For if this point is a pole of order k, there will exist a polynomial $P_{k-1}(z)$ such that $a_n = P_{k-1}(n)$, $n = 1, 2, \cdots$. Then $g(n) - P_{k-1}(n) = 0$, $n = 1, 2, \cdots$. But the function $g_2(z) = g(z) - P_{k-1}(z)$ satisfies the hypothesis of the theorem. Hence, as we have just seen, it cannot vanish at the points $1, 2, \cdots$.

Returning now to the proof of the main part of the theorem, we consider the functions

$$f_0(x) = \frac{x}{1-x} = \sum_{m=1}^{\infty} x^m,$$

$$f_1(x) = H_x^{(1)} f_0(x) = \sum_{m=1}^{\infty} m x^m,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$f_n(x) = H_x^{(n)} f_0(x) = \sum_{m=1}^{\infty} m^n x^m.$$

We shall prove that for each n there exist n positive numbers $b_p^{(1)}, b_p^{(2)}, \cdots, b_p^{(n)}$ such that

$$f_n(x) = \sum_{p=1}^{n} x^p b_p^{(n)} f_0^{(p)}(x), \tag{1}$$

and

$$b_n^{(n)} = 1, \; b_p^{(n)} \leq \frac{2^n (n-1)!}{p!}, \; p \leq n. \tag{2}$$

Suppose (1) is true for n. We have

$$f_{n+1}(x) = x f'_n(x),$$

$$f'_n(x) = \sum_{p=1}^{n} b_p^{(n)} [p x^{p-1} f_0^{(p)}(x) + x^p f_0^{(p+1)}(x)].$$

Hence

$$f_{n+1}(x) = \sum_{p=1}^{n} b_p^{(n)} [p x^p f_0^{(p)}(x) + x^{p+1} f_0^{(p+1)}(x)],$$

and if for convenience we write $b_0^{(n)} = 0$, we have the recurrence formulas

$$b_p^{(n+1)} = b_p^{(n)} p + b_{p-1}^{(n)}, \; p = 1, 2, \cdots, n,$$

$$b_{n+1}^{(n+1)} = 1.$$

Therefore (1) is true for $n+1$.

If (2) is valid for n, we have

$$b_p^{(n+1)} \leq \frac{2^n(n-1)!}{p!}p + \frac{2^n(n-1)!}{(p-1)!}, \; p \leq n+1$$
$$= \frac{2^{n+1}(n-1)!}{(p-1)!}$$
$$\leq \frac{2^{n+1}n!}{p!}.$$

Hence (2) holds for $n+1$.

Each $f_n(x)$ has the point $x = 1$ as its only singularity. Let us determine an upper bound for these functions in the region exterior to a circle with center at $x = 1$ and of arbitrary radius.

$$\left| x^p f_0^{(p)}(x) \right| = \left| x^p \frac{d^p}{dx^p} \frac{x}{1-x} \right| < p! \frac{|x|^p}{|1-x|^{p+1}}.$$

$$\left| f_n(x) \right| = \left| \sum_{p=1}^{n} x^p b_p^{(n)} f_0^{(p)}(x) \right|$$
$$< 2^n(n-1)! \sum_{p=1}^{n} \frac{|x|^p}{|1-x|^{p+1}}.$$

For x outside the circle, suppose $\left| 1-x \right| > h$, $\left| \frac{x}{1-x} \right| < M$, a constant greater than 1. Then

$$\frac{|x|^p}{|1-x|^{p+1}} < \frac{M^n}{|1-x|},$$

$$\sum_{p=1}^{n} \frac{|x|^p}{|1-x|^{p+1}} < \frac{nM^n}{|1-x|},$$

$$2^n(n-1)! \sum_{p=1}^{n} \frac{|x|^p}{|1-x|^{p+1}} < \frac{2^n n!\, M^n}{|1-x|},$$

$$|f_n(x)| < n!\, H^n,$$

where H is a constant.

It was proved by Poincaré that if $g(z) = \Sigma\alpha_n z^n$ is an integral function which satisfies the hypothesis of this theorem, then $\Sigma n!\,\alpha_n z^n$ is also an integral function. We have $|f_n(x)| < n!\ H^n$, and since $\Sigma n!|\alpha_n|\ |z|^n$ converges for all z, the series $\Sigma n!\ |\alpha_n|\ H^n$ converges. Hence $\Sigma\ |\alpha_n f_n(x)|$ converges, and the convergence is uniform outside the circle. In this region, the function

$$F(x) = \sum_{n=0}^{\infty} \alpha_n f_n(x)$$

is accordingly holomorphic.

Consider the sequence $\{S_m(x)\}$,

$$S_m(x) = \alpha_0 f_0(x) + \cdots + \alpha_m f_m(x).$$

We have seen that $\lim\limits_{m\to\infty} S_m(x)$ is a function having its only possible singularity at $x = 1$, since $S_m(x)$ converges uniformly to $\Sigma\alpha_n f_n(x)$ in every region exterior to a circle with center at $x = 1$ and of radius arbitrarily small. Hence the sequence $S_m^{(k)}(x)$ converges uniformly in the same region to $\dfrac{d^k}{dx^k}\sum\limits_{n=0}^{\infty} \alpha_n f_n(x)$. We have

$$\frac{d^k}{dx^k}\alpha_m f_m(x)\ \Big|_{x=0} = \alpha_m k^m k\,!,$$

$$S_m^{(k)}(0) = k\,!\sum_{i=0}^{m}\alpha_i k^i.$$

Hence $\lim S_m^{(k)}(x)\ |_{x=0} = k\,!\ g(k)$ is the k-th derivative of $\sum\limits_{n=0}^{\infty}\alpha_n f_n(x)$ at $x = 0$. The function $\sum\limits_{n=0}^{\infty}\alpha_n f_n(x)$, being regular for $|\,x\,| < 1$, may be developed in a Taylor's series, and the coefficient of order k is therefore $g(k)$. Hence we have

$$\begin{aligned} f(x) &= f(0) + \sum_{n=1}^{\infty} g(n)x^n \\ &= F(x) + f(0) - g(0), \end{aligned}$$

and therefore $f(x)$ has the point $x = 1$ as its only possible singularity. Thus the sufficiency of the condition is proved.

To prove the condition necessary, we note first that since $x = 1$ is the only singularity, $f(x)$ may be regarded as an integral function of the variable $\xi = \dfrac{1}{1-x}$, and therefore as an integral function of $\xi - 1 = \dfrac{x}{1-x}$:

$$f(x) = \sum_{n=0}^{\infty} A_n \left(\frac{x}{1-x}\right)^n, \quad \lim_{n\to\infty} \sqrt[n]{|A_n|} = 0.$$

Expand each term in a Taylor's series:

$$\left(\frac{x}{1-x}\right)^n = \sum_{k=0}^{\infty} C_{n+k-1}^{k} x^{n+k}, \quad |x| < 1$$

$$= \sum_{m=n}^{\infty} C_{m-1}^{m-n} \, x^m,$$

where, for convenience, $C_{-1}^{m} = 0$, $m > 0$, and $C_{-1}^{0} = 1$. Consequently

$$f(x) = \sum_{n=0}^{\infty} A_n \sum_{m=n}^{\infty} C_{m-1}^{m-n} x^m$$

$$= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} A_n \, C_{m-1}^{m-n} \right) x^m.$$

Hence we have

$$a_m = \sum_{n=0}^{m} A_n C_{m-1}^{m-n},$$

and, for $m \geq 1$,

$$a_m = A_1 + A_2(m-1) + A_3 \frac{(m-1)(m-2)}{2!} + \cdots + A_m. \tag{2}$$

We take now as the function $g(z)$ the following:

$$\begin{aligned} g(z) = A_1 + A_2(z-1) + \cdots & \\ + A_m \frac{(z-1)(z-2)\cdots(z-m+1)}{(m-1)!} & \\ + R_m(z), & \qquad (3) \end{aligned}$$

and show first that it is an integral function. Let r be an arbitrary positive number. We must show that the series for $g(z)$ converges uniformly in the circle of radius r. Since $\lim_{n\to\infty} \sqrt[n]{|A_n|} = 0$, we have

$$|A_m| < \epsilon^m \text{ for } m > m_0.$$

Hence, for $m > m_0$,

$$|R_m(z)| < \epsilon \left[\frac{(r+1)\cdots\cdots(r+m)}{m!}\epsilon^m + \frac{(r+1)\cdots\cdots(r+m+1)}{(m+1)!}\epsilon^{m+1} + \cdots\cdots\right].$$

This expression is the remainder after $m + 1$ terms of the development of $\dfrac{1}{(1-\epsilon)^{r+1}}$; hence its limit as m becomes infinite is zero, since ϵ may be taken less than 1. The series (3) accordingly represents an integral function.

Moreover, for any ϵ, $0 < \epsilon < 1$, $|g(z)| < e^{r^\epsilon}$ for $r > r_\epsilon$. For

$$\log|R_m(z)| < \log \epsilon + \log \left[\frac{(r+1)\cdots\cdots(r+m)}{m!}\epsilon^m + \frac{(r+1)\cdots\cdots(r+m+1)}{(m+1)!}\epsilon^{m+1} + \cdots\cdots\right]$$

$$< \log \epsilon + \log \frac{1}{(1-\epsilon)^{r+1}}$$

$$< \log \frac{1}{(1-\epsilon)^{r+1}}.$$

$$|R_m(z)| < e^{-(r+1)\log(1-\epsilon)},$$

$$|g(z)| < |P_m(z)| + e^{-(r+1)\log(1-\epsilon)},$$

where $P_m(z)$ is a polynomial of degree m.

Now corresponding to an arbitrary $\eta > 0$, the number ϵ may be taken so small that $|\log(1-\epsilon)| < \frac{\eta}{3}$. Then

$$|g(z)| < |P_m(z)| + e^{\frac{\eta}{3}(r+1)}$$

But $|P_m(z)| < Mr^m$ for all r. Moreover, $Mr^m < e^{\frac{r\eta}{3}}$ for r sufficiently large. Hence

$$\begin{aligned}|g(z)| &< e^{\frac{\eta r}{3}} + e^{\frac{\eta r}{3}} e^{\frac{\eta}{3}}\\ &< e^{\eta r}\end{aligned}$$

From (2) and (3) we have $g(n) = a_n$. The proof is therefore complete.[1]

41. The following theorem, whose proof is omitted, is due to Le Roy and Lindelöf:

THEOREM 2: *Let $g(z)$ be holomorphic in a certain half plane. If we have, for $-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$ and $r > r_\epsilon$,*

$$|g(\alpha + re^{\psi i})| < e^{(\theta+\epsilon)r},$$

where $\theta_0 < \pi$ is a constant, and $z - \alpha = re^{\psi i}$, then $f(x) = \Sigma g(n)x^n$ is holomorphic for all x in the sector $\theta_0 < \theta < 2\pi - \theta_0$, where $x = \rho e^{i\theta}$.

In particular the conclusion is verified if $g(z)$ is an integral function such that

$$|g(z)| < e^{(\theta_0+\epsilon)r} \text{ for } r > r_\epsilon, \tag{5}$$

the inequality being true for all ψ. The converse of this statement (with a hypothesis somewhat more general) was proved by the author for the case in which $f(x)$ has on the circle of convergence only a finite number of singularities on the arc $\theta_0 < \theta < 2\pi - \theta_0$. Then $f(x)$ may be represented as $\Sigma a_n x^n$ where $a_n = g(n)$, and $g(n)$ satisfies the inequality (5).

[1] The fact that two integral functions $g_1(z)$, $g_2(z)$, with $|g_i(z)| < e^{\epsilon r}$, can not be found so as to be the same at all of the values $z = 0, 1, 2, \ldots$ except a finite number (see § 42) implies that the $g(z)$ of the theorem can not in general take on the value a_0 when $z = 0$. [EDITOR.]